Can We Allow High Correlations in the Dictionary in the Synthesis Framework?

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I. EXTENDED ABSTRACT

Recovering a sparse signal from a given set of linear measurements has been a major subject of research in the recent decade. In the basic setup, an unknown signal \( x \in \mathbb{R}^d \) passes through a given linear transformation \( M \in \mathbb{R}^{m \times d} \) with an additive noise \( e \in \mathbb{R}^m \) providing a set of linear measurements \( y = Mx + e \). The signal \( x \) is assumed to have a sparse representation \( \alpha \in \mathbb{R}^n \) under a given dictionary \( D \in \mathbb{R}^{d \times n} \), i.e. \( x = Dx \). Most of the existing work dealing with the problem of recovering \( x \) from \( y \) focuses on the reconstruction of the signal’s representation, assuming that this leads to the desired signal recovery. This approach forces the dictionary to have a low coherence, and in particular, no linear dependencies between small groups of its columns.

Recently, a series of papers have shown that such dependencies can be allowed by aiming at recovering the signal itself \([1], [2], [3], [4], [5]\). However, these contributions consider the analysis modeling framework. Indeed, in \([3], [4], [5]\) it is even suggested that such linear dependencies are a virtue, and should be encouraged. Is this something unique for the analysis model, or does the same apply for the synthesis? A partial answer to this question is given in \([1]\), where the reconstruction conditions are presented in terms of the D-RIP, which is a property of the measurement matrix \( M \) for the synthesis model. However, as indicated in \([4]\), the results in \([1]\) essentially hold true for emerging signals from the analysis model.

Two recent contributions show that linear dependencies are permitted even in the synthesis context, as long as the aim of the pursuit is recovery of the signal (and not the representation) \([6], [7]\). The work in \([6]\) presents a modified version of CoSaMP that aims at recovering the signal, showing empirically that unlike the regular CoSaMP, the modified version gets a good recovery even in the presence of linear dependencies in \( D \). However, the theoretical part in this work assumes the existence of a near-optimal projection, like in \([4]\), which seems to be a NP-hard problem for a general dictionary. In \([7]\), the basic synthesis \( \ell_0 \)-minimization problem is analyzed, showing that in the presence of linear dependencies and high correlations in a dictionary, though the recovery of the representation is impossible, the signal reconstruction is doable. Both \([6]\) and \([7]\) rely on a modification of the restricted isometry property (RIP), the D-RIP \([1]\). However, both need a combinatorial computation for their results to hold.

In this work we also focus on the synthesis model and ask the same question: whether it is possible to have a general recovery guarantee for the synthesis model for a dictionary allowing linear dependencies between its columns. In addition, we ask the question, what happens when the columns are only nearly correlated. We seek theoretical guarantees that do not require a combinatorial computation.

We propose and analyze modified versions of the orthogonal matching pursuit (OMP) \([8]\) and iterative hard thresholding (IHT) \([9]\). Instead of using the D-RIP, we rely on two new properties of \( M \) and \( D \): The \( \epsilon \)-coherence and the \( \epsilon \)-RIP, which generalize the definitions of the regular coherence and RIP. These new definitions, \( T \) denotes a support set, \( \mathbf{d}_i \), the \( i \)-th column of \( D \), \( |T| \) the size of \( T \), and \( D_T \), a submatrix of \( D \) with columns corresponding to the indices in \( T \).

**Definition 1.1:** Let \( 0 \leq \epsilon < 1 \), \( M \) be a fixed measurement matrix and \( D \) a fixed dictionary. The \( \epsilon \)-coherence \( \mu_\epsilon (M,D) \) is defined as

\[
\mu_\epsilon(M,D) = \max_{i < j} \frac{|\langle M\mathbf{d}_i, M\mathbf{d}_j \rangle |}{\| M\mathbf{d}_i \|_2 \| M\mathbf{d}_j \|_2}.
\]

**Definition 1.2:** Let \( 0 \leq \epsilon \leq 1 \), \( M \) be a fixed measurement matrix and \( D \) be a fixed dictionary. The \( \epsilon \)-RIP \( \delta_\epsilon,\epsilon (M,D) \) is defined as the smallest value such that \( \forall z \in \mathbb{R}^{|T|} \) and \( \forall T \) for which \( |T| \leq k \) and \( \forall i \in T \), \( \| M(\mathbf{I} - D_T D_T^\dagger)\mathbf{d}_i \|_2 < \epsilon \| \mathbf{d}_i \|_2 \),

\[
(1 - \delta_\epsilon,\epsilon (M,D)) \| z \|_2^2 \leq \| M\mathbf{D}_T z \|_2^2 \leq (1 + \delta_\epsilon,\epsilon (M,D)) \| z \|_2^2.
\]

Note that for \( D = \mathbf{I} \) the \( \epsilon \)-coherence and \( \epsilon \)-RIP coincide with the regular coherence and RIP. Using these definitions we show that if \( k \leq \frac{1}{2}(1 + \frac{1}{\epsilon^2}) - O(\epsilon) \) or \( \delta_{k+1} \leq \frac{1}{2}(1 + O(\epsilon)) \) then the OMP signal recovery error is \( O(\epsilon) \). A similar result is presented for IHT.

These results imply that both methods can have an exact reconstruction in the case of linear dependencies within the dictionary columns. In the case of very high correlations, an almost exact recovery is guaranteed.

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REFERENCES


