Compressive Harmonic Retrieval via Matrix Completion

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Abstract—We study the problem of recovering a mixture of \( r \) complex multi-dimensional sinusoids from a random set of time measurements, where the frequencies can assume any value in a unit disk. Addressing this problem is important in many applications such as imaging system, control, radar, and can help remedy the basis mismatch issue in compressed sensing. We develop a novel algorithm based on enhanced \( K \)-fold Hankel structure and nuclear norm minimization. Under certain incoherence condition, this paradigm admits accurate recovery from the order of \( r \log^2(n_1, n_2) \) samples. Additionally, our results extend to the more general problem of low-rank Hankel matrix completion.

I. PROBLEM FORMULATION

A large class of signals of interest entails a superposition of spikes in the continuous frequency domain, which arises in numerous applications including radar, imaging systems, wireless networks, etc. The resolution of signal acquisition devices is often limited by physical and hardware constraints, precluding sampling with desired resolution. It is of great interest to identify the underlying multi-dimensional frequencies with infinite precision from partial observations. This is known as the harmonic retrieval problem; its 1-D version has been recently addressed in [1].

For concreteness, we restrict our discussion here to 2-D frequency models. Consider an \( n_1 \times n_2 \) data matrix \( X = (x_{i,j})_{0 \leq i < n_1, 0 \leq j < n_2} \) that can be expressed as \( x_{i,j} = \sum_{l=1}^{r} d_l y_i^l z_j^l \), where \( y_i^l \) and \( z_j^l \) are independent random variables drawn from a complex normal distribution \( \mathcal{CN}(0,1) \). Consider a random set of measurements \( \{(f_1, f_2) \mid 1 \leq i \leq r \} \). Suppose that there exists a location set \( \Omega \) of size \( m \) such that \( x_{k,l} \) is observed iff \( (k, l) \in \Omega \). We are interested in perfectly recovering \( X \) from a small set of measurements.

Once the data matrix \( X \) is recovered, the underlying frequencies can be retrieved using conventional methods such as MEMP [2].

We first convert \( X \) to an enhanced form \( X_e \) as introduced in [2]

\[
X_e := \begin{bmatrix}
X_0 & X_1 & \cdots & X_{n_1-k_1} \\
X_1 & X_2 & \cdots & X_{n_1-k_2} \\
& & \ddots & \vdots \\
X_{k_1-1} & X_{k_1} & \cdots & X_{n_1-1}
\end{bmatrix},
\]

where each block \( X_l \) is a Hankel matrix defined as

\[
X_l := \begin{bmatrix}
x_{l,0} & x_{l,1} & \cdots & x_{l,n_2-k_2} \\
x_{l,1} & x_{l,2} & \cdots & x_{l,n_2-k_2+1} \\
& & \ddots & \vdots \\
x_{l,k_2-1} & x_{l,k_2} & \cdots & x_{l,n_2-1}
\end{bmatrix}.
\]

This enhanced form admits a low-rank decomposition

\[
X_e = E_L D E_R,
\]

where \( D := \text{diag}(\{d_i \mid 1 \leq i \leq r \}) \). This reveals that \( X_e \) is low rank, i.e. \( \text{rank}(X_e) \leq r \). Besides, we denote by \( X_e \) the SVD of the enhanced form \( X_e = U \Lambda V^* \).

The enhanced matrix completion (EMaC) algorithm can be presented in the following semidefinite program:

\[
\begin{aligned}
\text{minimize} & \quad \|M_e\|_* \quad \text{subject to} \quad P_{12}(M) = P_{12}(X), \\
& \quad M_{e} \in \mathbb{R}^{n_1 \times n_2},
\end{aligned}
\]

where \( M_e \) denotes the enhanced form of a matrix \( M \).

II. PERFORMANCE GUARANTEE

We define a measure of incoherence as follows. Let \( G_L \) and \( G_R \) be two \( r \times r \) matrices such that

\[
\begin{aligned}
(G_L)_{ij} := \frac{1}{k_1 k_2} - \frac{1}{1-z_i^* z_j}, \\
(G_R)_{ij} := \frac{1}{(n_1-k_1+1)(n_2-k_2+1)} - \frac{1}{1-z_i^* z_j},
\end{aligned}
\]

with the convention that \( (G_L)_{ii} = (G_R)_{ii} = 1 \). Let \( \Omega_k(i, l) \) be the set of locations in the enhanced matrix \( X_e \) containing copies of elements \( x_{k,l} \), and denote \( \omega_k(i, l) = |\Omega_k(i, l)| \). Besides, we define \( A_{(i, j)} \) such that \( (A_{(i, j)})_{a, \beta} = 1/\sqrt{\omega_k(i, l)} \) if \( (a, \beta) \in \Omega_k(i, l) \) and 0 otherwise.

The incoherence can then be defined as follows.

Definition 1 (Incoherence). \( X \) is said to have incoherence \((\mu_1, \mu_2, \mu_3)\) if they are respectively the smallest values obeying

\[
\begin{align}
\sigma_{\min}(G_L) & \geq \frac{1}{\mu_1}, \quad \sigma_{\min}(G_R) \geq \frac{1}{\mu_2}; \\
\max_{(i, j) \in [n_1] \times [n_2]} |(UV^*)_{i,j}^2 | & \leq \frac{\mu_3 \sigma}{n_1 n_2}; \\
\max_{b \in [n_1] \times [n_2]} \sum_{a \in [n_1] \times [n_2]} |(UU^* A_0 V V^* \sqrt{\omega_k})_{a, \beta}^2 | & \leq \frac{\mu_3 \sigma}{n_1 n_2 \omega_k}.
\end{align}
\]

Theorem 1. Define \( c_r := \max \left( \frac{n_1 n_2}{k_1 k_2}, \frac{n_1 n_2}{n_1 k_2 + 1} \right) \). Then there exist constants \( c_0, c_0 > 0 \) such that under either of the following conditions:

1) Condition (3), (4), and (5) hold and \( m > c_0 \max(\mu_1 c_1, \mu_4, \mu_2 \sigma) \log^2(n_1 n_2) \);
2) Condition (3) holds and \( m > c_0 \mu_4 \sigma^2 \log^2(n_1 n_2) \).

\( X \) is the unique minimizer of EMaC with high probability.

III. NUMERICAL RESULTS

Set \( n_1 = n_2 = 15 \). We run 100 Monte Carlo trials for each triple \((n, m, r)\) when the frequencies are randomly generated. A trial is declared successful if the relative error \( \|X^* - X\|_F / \|X\|_F \leq 10^{-5} \). The empirical success probability is plotted in Fig. 1, which illustrates the practical ability of EMaC.

![Phase transition diagram of EMaC](image)

Figure 1. Phase transition diagram of EMaC when \( n_1 = n_2 = 15 \).

REFERENCES