Tensor Decompositions Tools for Multidimensional CS

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Abstract—Tensor decomposition models for multidimensional datasets have a long history in Mathematics and applied sciences. While these models have recently been applied to multidimensional signal processing, they were developed independently of the theory of sparse representations and Compressed Sensing (CS). We discuss, and illustrate recent results revealing connections among tensor decompositions models, recovery of low-rank multidimensional signals and CS theory.

I. INTRODUCTION

The study of the mathematical properties of multi-way arrays or tensors has started long time ago with the aim to generalize well known properties of the matrix case (N = 2) to higher number of dimensions (N ≥ 3). For example, the celebrated Singular Value Decomposition (SVD) and the rank of a matrix were successfully generalized to tensors in different ways by the Canonical Polyadic Decomposition (CPD) model and the Tucker model [2]. It is interesting to note that some surprising new properties arise when the number of dimensions is N ≥ 3. In the Tucker model, a tensor $\mathbf{Y} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ is approximated by the following factorization:

$$\mathbf{Y} \approx \mathbf{X} \times_1 \mathbf{D}_1 \times_2 \cdots \times_N \mathbf{D}_N; \quad (1)$$

where $\mathbf{X} \in \mathbb{R}^{I_1 \times \cdots \times I_N}$ is the “core tensor” and matrices $\mathbf{D}_n \in \mathbb{R}^{I_n \times R_n}$ (n = 1, 2, ..., N) are “factors”. The mode-n “tensor by matrix” product $\mathbf{Y} = \mathbf{X} \times_n \mathbf{D}$ is defined by $y_{i_1 ... i_n} = \sum_{i_{n+1} ... i_N} x_{i_1 ... i_{n-1} i_{n+1} ... i_N} d_{j_{n+1}...j_N}$. We define mode-n fibers as the vectors obtained by fixing all indices except one in mode n, i.e. columns and rows are mode-1 and mode-2 fibers of a matrix, respectively. By collecting all mode-n fibers of a tensor as columns of a matrix we obtain the so-called unfolding matrix which is noted as $\mathbf{Y}^{(n)} \in \mathbb{R}^{I_1 \times \cdots \times I_{n-1} \times I_{n+1} \times \cdots \times I_N}$. A tensor is said to have a rank-(R_1, R_2, ..., R_N) Tucker representation if (R_1, R_2, ..., R_N) are the minimum values for which eq. (1) holds exactly. It is noted that $R_n$ corresponds to the rank of the unfolding matrix $\mathbf{Y}^{(n)}$.

A. Reconstruction of low rank Tucker tensors

If a tensor $\mathbf{Y} \in \mathbb{R}^{I_1 \times I_2 \times \cdots \times I_N}$ has an exact rank-(R_1, R_2, ..., R_N) Tucker representation its mode-n fibers belong to an $R_n$-dimensional linear subspace, so a set of $R_n$ fibers do exists that span the columns of $\mathbf{Y}^{(n)}$ [2]. In fact, in [1], a closed formula was provided that allows one to reconstruct a whole tensor of rank-(R_1, R_2, ..., R_N) from a small subset of fibers which is called Fiber Sampling Tensor Decomposition (FSTD). FSTD determines an intersection sub-tensor $\mathbf{W} \in \mathbb{R}^{R_1 \times R_2 \times \cdots \times R_N}$ by selecting $R_n$ indices in each mode so that the following exact Tucker representation is obtained:

$$\mathbf{Y} = \mathbf{W} \times_1 \mathbf{C}_1 \mathbf{W}_1^T \times_2 \cdots \times_N \mathbf{C}_N \mathbf{W}_N^T; \quad (2)$$

where matrices $\mathbf{C}_n \in \mathbb{R}^{R_n \times I_1 \cdots R_{n-1} \cdot I_{n+1} \cdots I_N}$ comprise the mode-n fibers defined through such index restrictions in modes $m \neq n$. This formula can be used as a low rank approximation of a multidimensional dataset (Fig 1. (a)). FSTD can be used also for multidimensional CS if it is applied to fibers obtained as random projections in each mode which is currently under study.

B. Sparse representation of tensors

It is known that multidimensional signals can be sparsely represented by using Kronecker dictionaries. Recently, some authors have studied their properties (see for example [3]). In fact, using the Kronecker structure is equivalent to adopt the Tucker tensor representation of eq. (1) where factors $\mathbf{D}_n$ play the role of mode-n dictionaries and $\mathbf{X}$ is a sparse tensor of coefficients (in this case we usually use $R_n \geq I_n$) [4]. Thus, CS problems are reduced to solve a large underdetermined systems of algebraic equations with Kronecker structure and sparsity constraints. An efficient greedy algorithm, called N-BOMP [4] is obtained by exploiting the Kronecker structure combined with a block-sparsity constraint on the tensor of coefficients $\mathbf{X}$ (Fig 1. (b)). It is interesting to note that a better theoretical guarantee of the OMP approach based on coherence is obtained when tensor block-sparsity is taken into account [4]. Other problems, such as tensor completion can be approached by exploiting the Kronecker structure of dictionaries applied to overlapped tensor-patches which comprises to solve underdetermined linear equations with Katri-Rao structure as it was recently proposed in [5] (Fig 1. (c)).

REFERENCES