The Computational Complexity of Spark, RIP, and NSP

Andreas M. Tillmann and Marc E. Pfetsch
Research Group Optimization, Technische Universität Darmstadt
Dolivostr. 15, 64293 Darmstadt, Germany

Abstract—Over the past years, several conditions were introduced which guarantee that the NP-hard problem of finding the sparsest (exact or approximate) solution to an underdetermined linear system can be solved by efficient algorithms. The most well-known ones are the mutual coherence, the restricted isometry property (RIP), and the nullspace property (NSP). While evaluating the mutual coherence of a given matrix is easy, it has been suspected for some time that evaluating RIP and NSP is computationally intractable. We confirm these conjectures by showing that for a given matrix and positive integer order, computing the best constants for which the RIP or NSP (of this order) hold is, in general, NP-hard in the strong sense. These results are based on the fact that determining the spark of a matrix is strongly NP-hard, which we also present in this work.

I. INTRODUCTION

The task of finding a sparsest solution to an underdetermined linear system, i.e.,

\[
\min \|x\|_0 \quad \text{s.t.} \quad Ax = b, \tag{P_0}
\]

is well-known to be (strongly) NP-hard (cf. [MP5] in [1]); the same is true for the variant with \(Ax = b\) replaced by \(\|Ax - b\|_2 \leq \varepsilon\). Thus, in practice, one often resorts to heuristics. One of the most popular approaches is known as basis pursuit (BP) or \(\ell_1\)-minimization, where instead of (P_0) one considers

\[
\min \|x\|_1 \quad \text{s.t.} \quad Ax = b. \tag{P_1}
\]

Under certain conditions, the optimal solutions of (P_0) and (P_1) are unique and coincide; one says that \(\ell_0-\ell_1\)-equivalence holds or that the \(\ell_0\)-solution can be recovered by \(\ell_1\)-minimization. Similar results exist for the sparse approximation (denoising) variants with constraint \(\|Ax - b\|_2 \leq \varepsilon\) and other related problems.

Many such conditions employ the famous restricted isometry property (RIP) [2], which is satisfied with order \(k\) and a constant \(\delta_k\) if a given matrix \(A\) satisfies

\[
(1 - \delta_k)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1 + \delta_k)\|x\|_2^2 \tag{1}
\]

for all vectors \(x\) with \(1 \leq \|x\|_0 \leq k\). One is usually interested in the smallest possible constant \(\delta_k\), the restricted isometry constant (RIC), such that (1) is fulfilled. For instance, if \(\delta_{2k} \leq 1/2\), all \(x\) with at most \(k\) nonzero entries can be recovered from \((A, b) = (Ax)\) via basis pursuit. Several probabilistic results show that certain random matrices are highly likely to satisfy the RIP with desirable values of \(\delta_k\); there has also been work on deterministic matrix constructions aiming at relatively good RIPS. Moreover, the RIP provides sparse recovery guarantees for other heuristics such as Orthogonal Matching Pursuit and variants, as well as in the denoising case.

Another popular tool for guaranteeing \(\ell_0-\ell_1\)-equivalence is the nullspace property (NSP), see, e.g., [4], which characterizes recoverability by (P_1) (in fact, via \(\ell_p\)-minimization with \(0 < p \leq 1\)) for sufficiently sparse solutions of (P_0). The NSP of order \(k\) is satisfied with constant \(\alpha_k\) if

\[
\|x\|_{k,1} \leq \alpha_k \|x\|_1 \quad \tag{2}
\]

holds for all vectors \(x\) in the nullspace of \(A\) (i.e., \(Ax = 0\)), where \(\|x\|_{k,1}\) denotes the sum of the \(k\) largest absolute values of entries in \(x\). Similar to the RIP case, one is interested in the smallest constant \(\alpha_k\), the nullspace constant (NSC), such that (2) is fulfilled. Indeed, if and only if \(\alpha_k < 1/2\), (P_1) with \(b := Ax\) and \(\|x\|_0 \leq k\), has the unique solution \(x\), which coincides with that of \((P_0)\). Again, error bounds for recovery in the denoising case can be given as well.

II. COMPLEXITY OF SPARK, RIP, AND NSP

With the following results, we settle long-standing conjectures about the computational intractability of evaluating the RIP and NSP. \(\square\)

Theorem 1: Given a matrix \(A \in \mathbb{Q}^{m \times n}\) and a positive integer \(k\), the problem to decide whether there exists some constant \(\delta_k < 1\) such that \(A\) satisfies the RIP of order \(k\) with constant \(\delta_k\) is strongly coNP-complete. Consequently, it is strongly NP-hard to compute the RIP \(\delta_k\). Moreover, if \(\delta_k \in (0, 1)\) is also part of the given input, it is strongly NP-hard to certify whether \(A\) satisfies the RIP of order \(k\) with constant \(\delta_k\). (Note that, using the Turing complexity model, we need to work with \(\mathbb{Q}\), not \(\mathbb{R}\), in order to obtain finitely representable numbers; moreover, coNP is the complement of NP.)

Theorem 2: Given a matrix \(A \in \mathbb{Q}^{m \times n}\) and a positive integer \(k\), the problem to decide whether \(A\) satisfies the NSP of order \(k\) with some constant \(\alpha_k < 1\) is strongly coNP-complete. Consequently, it is strongly NP-hard to compute the NSC \(\alpha_k\).

The proofs are based on the following result about inclusion-wise minimal sets of linearly dependent columns, i.e., circuits, of \(A\). The smallest cardinality of a circuit of \(A\) is the girth of the vector matroid associated with \(A\); this number is also known as the spark of \(A\), i.e.,

\[
\text{spark}(A) := \min \{\|x\|_0 : Ax = 0, \ x \neq 0\}.
\]

Theorem 3: Given a matrix \(A \in \mathbb{Q}^{m \times n}\) and a positive integer \(k\), the problem to decide whether there exist a circuit of \(A\) of size at most \(k\) is NP-complete in the strong sense. Consequently, it is strongly NP-hard to compute \(\text{spark}(A)\).

Computing \(\text{spark}(A)\) was shown to be weakly NP-hard in [5], by reducing the Subset Sum Problem (cf. [MP9] in [1]) to the task of deciding whether the spark equals the number of rows \((m)\). Our Theorem 3 strengthens this result to NP-hardness in the strong sense, for \(k \leq m\), by a reduction from the \(k\)-Clique Problem (cf. [GT9] in [1]). The proof extends a result from [6] about the girth of transversal matroids to vector matroids. (Recall that strong NP-hardness implies that no fully polynomial-time approximation scheme (FPTAS) can exist, unless \(P = NP\).)

REFERENCES